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[3 : 2]-pairs of symmetric group algebras and their intermediate defect 4 blocks ☆

Stuart Martin [a,*], Kai Meng Tan [b]

[a] *DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, UK*
[b] *Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543*

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Abstract

In this paper we study [3 : 2]-pairs of symmetric group algebras and their 'intermediate' block in detail. The aim is to understand how one block of a [3 : 2]-pair can inherit two properties—the Ext-quiver being bipartite and the principal indecomposable modules having a common Loewy length 7—from the other. We establish some sufficient conditions for this inheritance, and verify these conditions for some special blocks.

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group on n letters, and k an algebraically closed field of characteristic p ($\geqslant 5$). In [10], we showed that the principal blocks of $k\mathfrak{S}_{3p+r}$ ($0 \leqslant r < p$) have the following common properties:

(P1) The Ext-quiver is bipartite.

☆ This work incorporates a revised version of part of the second author's doctoral thesis [K.M. Tan, Small defect blocks of symmetric group algebras, PhD thesis, University of Cambridge, 1998].

* Corresponding author.
E-mail addresses: sm@dpmms.cam.ac.uk (S. Martin), tankm@nus.edu.sg (K.M. Tan).

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(P2) The principal indecomposable modules have common Loewy length equal to 7.

We also studied in detail how a defect 3 block in a [3 : 1]-pair may inherit these two properties from the other, establishing some sufficient conditions for this to hold.

In this paper, we study [3 : 2]-pairs of symmetric group algebras and their 'intermediate' block in detail, with the aim of understanding how one block of a [3 : 2]-pair can inherit properties (P1) and (P2) from the other.

Our approach is as follows. We begin by giving a short account of the representation theory we require. In Section 3, we present the known background on [3 : 2]-pairs as shown in [9]. In Section 4, we study the so-called 'intermediate blocks' of [3 : 2]-pairs, followed by obtaining new information about the [3 : 2]-pairs in Section 5. We are then able to obtain some sufficient conditions for the inheritance of properties (P1) and (P2) in Section 6. We conclude by verifying in Section 7 these sufficient conditions for the defect 3 blocks B_i $(1 \leqslant i \leqslant p)$ having p-core $(p+i-2, i-1)$, thus showing that (P1) and (P2) hold for these blocks.

2. Preliminaries

In this section, we give a brief account of the representation theory which we require. For more detailed accounts, we refer the reader to [2,3] for the representation theory of symmetric groups, and to [8] for general theory of group representations.

Firstly, the following notations will be used in this paper:

(1) the projective cover of a module M will be denoted by $P(M)$, and $\Omega(M)$ will denote the submodule of $P(M)$ satisfying $P(M)/\Omega(M) \cong M$;
(2) a filtration $M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_r = 0$ will be denoted by a matrix with r rows, where the ith row is the factor M_{i-1}/M_i;
(3) the multiplicity of a simple module S as a composition factor of a module M will be denoted by $[M : S]$;
(4) $M \sim M'$ means the two modules M and M' have the same composition factors (with multiplicities).

The following easy lemma on general modules will be used often in this paper.

Lemma 2.1. *Suppose M is a module having a composition series, and N is a submodule of M such that $N/\operatorname{rad}(N) \cong nS$ for some simple module S. If L is a submodule of M such that $[L : S] = [M : S]$, then $N \subseteq L$.*

In particular, if $[M : S] = [N : S]$, then N contains (as a subset) every submodule K of M such that $K/\operatorname{rad}(K) \cong kS$.

Proof. If $N \nsubseteq L$, then $1 \leqslant [N/(N \cap L) : S] = [(N+L)/L : S] \leqslant [M/L : S] = 0$, a contradiction. $\square$

Recall that the Specht modules S^λ, as λ runs through the set of partitions of n, give a complete list of mutually non-isomorphic simple modules of $\mathfrak{S}_n$ in zero characteristic. In positive characteristic p, the Specht module S^λ has a simple head D^λ if the partition λ is p-regular; all composition factors D^μ of its radical satisfy $\mu \rhd \lambda$. If λ is p-singular, then all composition factors D^μ of S^λ satisfy $\mu \rhd \lambda$. As λ runs through the p-regular partitions, the set of simple heads is a complete list of mutually non-isomorphic simple modules of $k\mathfrak{S}_n$.

Two Specht modules S^λ and S^μ of $k\mathfrak{S}_n$ lie in the same block if, and only if, λ and μ have the same p-core (Nakayama's Conjecture). The Branching Rule provides a Specht filtration for the restricted Specht module $S^\lambda{\downarrow_{\mathfrak{S}_{n-1}}}$ and the induced Specht module $S^\lambda{\uparrow^{\mathfrak{S}_{n+1}}}$. The Specht module S^μ is a factor in this filtration if, and only if, μ can be obtained from λ by removing or adding a node. A factor S^α lies above another factor S^β in this filtration if $\alpha \rhd \beta$.

The Ext-quiver of a k-algebra A is a finite directed graph whose vertex set is labelled by the (isomorphism classes of) simple A-modules and the number of edges from S_1 to S_2 is given by $\dim_k \operatorname{Ext}^1_A(S_1, S_2)$. Since the simple modules of symmetric group algebras are self-dual, all edges in its Ext-quiver are directed two-ways. Recall that a (directed) graph is termed bipartite if there is a partition of its vertex set into two parts such that there is no edge between any two vertices in the same part. If the Ext-quiver of A is bipartite, then one of the consequences is that if $P(S)$ is the projective cover of a simple A-module and S' is a simple module occurring in an odd (respectively even) Loewy layer of $P(S)$, then all composition factors of $P(S)$ isomorphic to S' lie in odd (respectively even) Loewy layer(s) of $P(S)$.

A key ingredient in our presentation is Kleshchev's work [5–7] on the restricted simple module $D^\lambda{\downarrow_{\widetilde{B}}}$. Let B be a block of $k\mathfrak{S}_n$ and let $\widetilde{B}$ be a block of $k\mathfrak{S}_{n-1}$ such that the core of the latter has an abacus display having the same number of beads on each column as that of B, except the $(i-1)$th and ith columns, where respectively there is one bead more than and one bead less than that of B. Let λ be a p-regular partition of B. A bead lying on the ith column and jth row of the abacus display of λ is *normal* if the position on its left is unoccupied and, for all $l > j$, counting the beads between the $(j+1)$th and lth row (both inclusive), those lying on the $(i-1)$th column is not more than those lying on the ith column. A normal bead is *good* if it is the highest (in the abacus display) normal bead. We may move a normal bead of λ one position to its left and obtain a p-regular partition $\tilde{\lambda}$ of $\widetilde{B}$. The multiplicity of the simple module $D^{\tilde{\lambda}}$ as a composition factor of $D^\lambda{\downarrow_{\widetilde{B}}}$ is one more than the number of normal beads below the normal bead moved to obtain $D^{\tilde{\lambda}}$. If the normal bead moved is good, then the simple module $D^{\tilde{\lambda}}$ obtained is the head and socle of $D^\lambda{\downarrow_{\widetilde{B}}}$. In particular, $D^\lambda{\downarrow_{\widetilde{B}}}$ is either zero (when there is no normal bead for λ), or has a simple head and a simple socle (and thus indecomposable). Moreover, two non-isomorphic simple modules of $k\mathfrak{S}_n$ may not give non-zero isomorphic simple heads when restricted to $\widetilde{B}$.

Schaper [11] has a formula for calculating an upper bound of each entry in the decomposition matrix of a block of symmetric group algebras. In fact his upper bound is non-zero if and only if the entry is non-zero. If a symmetric group block has the property that each entry in its decomposition matrix is at most 1, then the formula determines the decomposition matrix completely. For one version of this formula suited to our method, we refer the reader to [9].

Let us recall also the non-standard notation used in [10] for weight 3 partitions having p-core τ.

Definition 2.2. Let τ be a p-core having r parts. Let b be a fixed integer not less than $r+3p$. Any weight 3 partition λ having core τ may be represented on the abacus having b beads. Then λ may be denoted using the $\langle\,\rangle$-*notation*, defined as follows: if the abacus display having b beads of λ has

(1) one bead of weight 3 on column i, then denote λ by $\langle i\rangle$;
(2) one bead of weight 2 on column i and one bead of weight 1 on column j, then denote λ by $\langle i, j\rangle$;
(3) three beads of weight 1 on column(s) i, j and k, then denote λ by $\langle i, j, k\rangle$.

It is clear that the $\langle\,\rangle$-notation depends on the number of beads used in the abacus display. There is usually a natural choice for the number of beads used. For example, if the p-core of the defect 3 block B has r parts, then we usually use the $\langle\,\rangle$-notation with $3p+r$ beads to denote the partitions of B.

3. Background

In this section we remind the reader of the results of Russell and the first author [9] on defect 3 blocks and, in particular, $[3:2]$-pairs.

In [9], a proof that the defect 3 blocks of symmetric group algebras have the following properties is announced.

(X1) Each entry of its decomposition matrix is either 0 or 1.
(X2) The Ext^1-space between any two simple modules is at most one-dimensional.
(X3) Every simple module does not extend itself.

However, James and Mathas [4] found gaps in the proof of property (X1), and as the proof of properties (X2) and (X3) relies on the validity of (X1), this renders the proof of these three properties incomplete.

In April 2004, Fayers [1] produced a complete proof of property (X1), thereby establishing the truth of [9].

Throughout this paper, we shall often make use of these three properties, sometimes without comment.

For the rest of this paper, B will be denote a defect 3 block of $k\mathfrak{S}_n$ which forms a $[3:2]$-pair with $\widetilde{B}$, a defect 3 block of $k\mathfrak{S}_{n-2}$, with the ith column of the abacus display of the core of B having two beads more than that of $\widetilde{B}$.

We begin by introducing some (non-standard) terminology.

Definition 3.1. With respect to the $[3:2]$-pair B and $\widetilde{B}$, we call

(1) a partition λ of B is *exceptional* if we can move more than two beads on the ith column of its abacus display to their respective preceding positions on the $(i-1)$th column. Otherwise, it is *non-exceptional*;
(2) a Specht module S^λ of B is *exceptional* if and only if λ is exceptional;
(3) a simple module D^λ of B is *exceptional* if $D^\lambda\downarrow_{\widetilde{B}}$ is not semisimple. Otherwise, it is *non-exceptional*;
(4) a partition $\tilde{\lambda}$ of $\widetilde{B}$ is *exceptional* if we can move more than two beads on the $(i-1)$th column of its abacus display to their respective succeeding positions on the ith column. Otherwise, it is *non-exceptional*;
(5) a Specht module $S^{\tilde{\lambda}}$ of $\widetilde{B}$ is *exceptional* if, and only if, $\tilde{\lambda}$ is exceptional;
(6) a simple module $D^{\tilde{\lambda}}$ of $\widetilde{B}$ is *exceptional* if $D^{\tilde{\lambda}}\uparrow^B$ is not semisimple. Otherwise, it is *non-exceptional*.

There are four exceptional Specht modules of B, denoted as S^α, S^β, S^γ and S^δ, whose corresponding partitions have the following $(i-1)$th and ith columns in their abacus displays:

$$
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
- & \bullet \\
- & \bullet \\
\multicolumn{2}{c}{\alpha}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
- & \bullet \\
- & \bullet \\
\multicolumn{2}{c}{\beta}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet \\
\bullet & - \\
- & \bullet \\
\multicolumn{2}{c}{\gamma}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet \\
- & \bullet \\
\bullet & - \\
\multicolumn{2}{c}{\delta}
\end{array}
$$

Similarly, there are four exceptional Specht modules of $\widetilde{B}$, denoted as $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$, whose corresponding partitions have the following $(i-1)$th and ith columns in their abacus displays:

$$
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
\bullet & - \\
\bullet & - \\
- & \bullet \\
\multicolumn{2}{c}{\widetilde{\alpha}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
\bullet & - \\
- & \bullet \\
\bullet & - \\
\multicolumn{2}{c}{\widetilde{\beta}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
\bullet & - \\
\bullet & - \\
\multicolumn{2}{c}{\widetilde{\gamma}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
\bullet & - \\
\bullet & - \\
\multicolumn{2}{c}{\widetilde{\delta}}
\end{array}
$$

The partitions α and $\widetilde{\alpha}$ are always p-regular. So are the conjugate partitions δ' and $\widetilde{\delta}'$. The diagrams below show the dependence of p-regularity among the exceptional partitions and their conjugates.

$$\begin{array}{ccccc}
\beta\ p\text{-regular} & \Longrightarrow & \gamma\ p\text{-regular} & \Longrightarrow & \delta\ p\text{-regular} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\widetilde{\delta}\ p\text{-regular} & \Longrightarrow & \widetilde{\gamma}\ p\text{-regular} & \Longrightarrow & \widetilde{\beta}\ p\text{-regular}
\end{array}$$

$$\begin{array}{ccccc}
\gamma'\ p\text{-regular} & \Longrightarrow & \beta'\ p\text{-regular} & \Longrightarrow & \alpha'\ p\text{-regular} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\widetilde{\alpha}'\ p\text{-regular} & \Longrightarrow & \widetilde{\beta}'\ p\text{-regular} & \Longrightarrow & \widetilde{\gamma}'\ p\text{-regular}
\end{array}$$

Restriction of the exceptional Specht modules of B and induction of the exceptional Specht modules of $\widetilde{B}$ give the following:

$$\begin{aligned}
&\text{(C1)} \quad S^{\alpha}{\downarrow_{\widetilde{B}}} \sim 2\big(S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}}\big);\\
&\text{(C2)} \quad S^{\beta}{\downarrow_{\widetilde{B}}} \sim 2\big(S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}} \oplus S^{\widetilde{\delta}}\big);\\
&\text{(C3)} \quad S^{\gamma}{\downarrow_{\widetilde{B}}} \sim 2\big(S^{\widetilde{\alpha}} \oplus S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}}\big);\\
&\text{(C4)} \quad S^{\delta}{\downarrow_{\widetilde{B}}} \sim 2\big(S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}}\big);\\
&\text{(D1)} \quad S^{\widetilde{\alpha}}{\uparrow^{B}} \sim 2\big(S^{\alpha} \oplus S^{\beta} \oplus S^{\gamma}\big);\\
&\text{(D2)} \quad S^{\widetilde{\beta}}{\uparrow^{B}} \sim 2\big(S^{\alpha} \oplus S^{\beta} \oplus S^{\delta}\big);\\
&\text{(D3)} \quad S^{\widetilde{\gamma}}{\uparrow^{B}} \sim 2\big(S^{\alpha} \oplus S^{\gamma} \oplus S^{\delta}\big);\\
&\text{(D4)} \quad S^{\widetilde{\delta}}{\uparrow^{B}} \sim 2\big(S^{\beta} \oplus S^{\gamma} \oplus S^{\delta}\big).
\end{aligned}$$

There is only one exceptional simple module of B, namely D^{α}. All other simple modules of B remain semisimple when restricted to $\widetilde{B}$. In fact, if D^{λ} is a non-exceptional simple module of B, then there is a unique simple module $D^{\tilde{\lambda}}$ of $\widetilde{B}$ such that $D^{\lambda}{\downarrow_{\widetilde{B}}} \cong 2D^{\tilde{\lambda}}$. Similarly, there is only one exceptional simple module of $\widetilde{B}$, namely $D^{\widetilde{\alpha}}$. All other simple modules of $\widetilde{B}$ remain semisimple when induced to B. If $D^{\widetilde{\mu}}$ is a non-exceptional simple module of $\widetilde{B}$, then there is a unique simple module D^{μ} of B such that $D^{\widetilde{\mu}}{\uparrow^{B}} \cong 2D^{\mu}$. The restricted module $D^{\alpha}{\downarrow_{\widetilde{B}}}$ has six copies of $D^{\widetilde{\alpha}}$, and only $D^{\widetilde{\alpha}}$ occurs in its head. Similarly, the induced module $D^{\widetilde{\alpha}}{\uparrow^{B}}$ has six copies of D^{α}, and only D^{α} occurs in its head.

The exceptional simple module D^{α} is a composition factor of S^{α}, S^{β}, S^{γ} and S^{δ}, and in no other Specht module is D^{α} a composition factor. The projective cover $P(D^{\alpha})$ has a Specht filtration

$$0 \subset M_3 \subset M_2 \subset M_1 \subset P(D^{\alpha})$$

such that $P(D^{\alpha})/M_1 \cong S^{\alpha}$, $M_1/M_2 \cong S^{\beta}$, $M_2/M_3 \cong S^{\gamma}$ and $M_3 \cong S^{\delta}$.

Similarly, $D^{\widetilde{\alpha}}$ is a composition factor of $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$, and in no other Specht module is it a composition factor. By analogy its projective cover has a similar Specht filtration.

We may also note the following.

(1) If β is p-regular (equivalent to $\widetilde{\delta}$ being p-regular), then $D^{\beta}\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\delta}}$ and $D^{\widetilde{\delta}}\uparrow^{B} \cong 2D^{\beta}$. Moreover, D^{β} occurs in both S^{γ} and S^{δ}.
(2) If γ is p-regular (equivalent to $\widetilde{\gamma}$ being p-regular), then $D^{\gamma}\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\gamma}}$ and $D^{\widetilde{\gamma}}\uparrow^{B} \cong 2D^{\gamma}$. Moreover, D^{γ} and $D^{\widetilde{\gamma}}$ occur in S^{δ} and $S^{\widetilde{\delta}}$, respectively.
(3) If δ is p-regular (equivalent to $\widetilde{\beta}$ being p-regular), then $D^{\delta}\downarrow_{\widetilde{B}} \cong 2D^{\widetilde{\beta}}$ and $D^{\widetilde{\beta}}\uparrow^{B} \cong 2D^{\delta}$. Moreover, $D^{\widetilde{\beta}}$ occur in both $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$.

4. The intermediate block $\overline{B}$

Given any partition of B, there are beads on the ith column of its abacus display which may be moved to their respective preceding positions on the $(i-1)$th column. Moving one of these beads corresponds to restricting the associated Specht module of B to a defect 4 block $\overline{B}$. The abacus display of the p-core of $\overline{B}$ has one bead more on the $(i-1)$th column and one bead less on the ith column than that of B. Similarly, moving one of the beads on the $(i-1)$th column in the abacus display of a partition of $\widetilde{B}$ to its succeeding position on the ith column corresponds to inducing the associated Specht module of $\widetilde{B}$ to $\overline{B}$. In this section, we study this defect 4 block $\overline{B}$ with the aim of understanding our $[3:2]$-pair better.

We first note that $\overline{B}$ is the 'intermediate' block between the blocks B and $\widetilde{B}$ when we restrict or induce modules of one to the other, in the following sense.

Lemma 4.1. *Let M be a B-module. Then $M\downarrow_{\widetilde{B}} \cong (M\downarrow_{\overline{B}})\downarrow_{\widetilde{B}}$. Similarly, if N is a $\widetilde{B}$-module, then $N\uparrow^{B} \cong (N\uparrow^{\overline{B}})\uparrow^{B}$.*

Proof. We know that $M\downarrow_{\mathfrak{S}_{n-2}} \cong (M\downarrow_{\mathfrak{S}_{n-1}})\downarrow_{\mathfrak{S}_{n-2}}$. Using the Branching Rule, we see that the summands of $M\downarrow_{\mathfrak{S}_{n-1}}$ which do not lie in $\overline{B}$ vanish when they are restricted to $\widetilde{B}$. Thus, $(M\downarrow_{\mathfrak{S}_{n-1}})\downarrow_{\widetilde{B}} \cong (M\downarrow_{\overline{B}})\downarrow_{\widetilde{B}}$. The first statement now follows easily. A similar argument applies for the second statement. $\square$

With exactly six exceptions, every Specht module of $\overline{B}$ restricts to a unique Specht module of $\widetilde{B}$ (or gives zero) and induces to a unique Specht module of B (or gives zero). The exceptional Specht modules will be denoted as $S^{\bar{\alpha}}$, $S^{\bar{\beta}}$, $S^{\bar{\gamma}}$, $S^{\bar{\delta}}$, $S^{\bar{\epsilon}}$ and $S^{\bar{\kappa}}$, and their corresponding partitions have the following $(i-1)$th and ith columns in the abacus display:

$$
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
\bullet & - \\
- & \bullet \\
- & \bullet \\
\multicolumn{2}{c}{\bar{\alpha}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
\bullet & - \\
- & \bullet \\
\multicolumn{2}{c}{\bar{\beta}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
\bullet & - \\
- & \bullet \\
\multicolumn{2}{c}{\bar{\gamma}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet & - \\
- & \bullet \\
- & \bullet \\
\bullet & - \\
\multicolumn{2}{c}{\bar{\delta}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
\bullet & - \\
- & \bullet \\
\bullet & - \\
\multicolumn{2}{c}{\bar{\epsilon}}
\end{array}
\qquad
\begin{array}{cc}
i-1 & i \\
\vdots & \vdots \\
\bullet & \bullet \\
- & \bullet \\
- & \bullet \\
\bullet & - \\
\bullet & - \\
\multicolumn{2}{c}{\bar{\kappa}}
\end{array}
$$

These Specht modules of $\overline{B}$ have the following relationships with the exceptional Specht modules of $\widetilde{B}$ and B:

$$
\begin{array}{lll}
\text{(E1)} & S^{\bar{\alpha}}{\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}}; & S^{\bar{\alpha}}{\uparrow}^{B} \sim S^{\alpha} \oplus S^{\beta}; \\
\text{(E2)} & S^{\bar{\beta}}{\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\gamma}}; & S^{\bar{\beta}}{\uparrow}^{B} \sim S^{\alpha} \oplus S^{\gamma}; \\
\text{(E3)} & S^{\bar{\gamma}}{\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\delta}}; & S^{\bar{\gamma}}{\uparrow}^{B} \sim S^{\beta} \oplus S^{\gamma}; \\
\text{(E4)} & S^{\bar{\delta}}{\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}}; & S^{\bar{\delta}}{\uparrow}^{B} \sim S^{\alpha} \oplus S^{\delta}; \\
\text{(E5)} & S^{\bar{\epsilon}}{\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\beta}} \oplus S^{\widetilde{\delta}}; & S^{\bar{\epsilon}}{\uparrow}^{B} \sim S^{\beta} \oplus S^{\delta}; \\
\text{(E6)} & S^{\bar{\kappa}}{\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}}; & S^{\bar{\kappa}}{\uparrow}^{B} \sim S^{\gamma} \oplus S^{\delta}; \\
\text{(F1)} & S^{\widetilde{\alpha}}{\uparrow}^{\overline{B}} \sim S^{\bar{\alpha}} \oplus S^{\bar{\beta}} \oplus S^{\bar{\gamma}}; & S^{\alpha}{\downarrow}_{\overline{B}} \sim S^{\bar{\alpha}} \oplus S^{\bar{\beta}} \oplus S^{\bar{\delta}}; \\
\text{(F2)} & S^{\widetilde{\beta}}{\uparrow}^{\overline{B}} \sim S^{\bar{\alpha}} \oplus S^{\bar{\delta}} \oplus S^{\bar{\epsilon}}; & S^{\beta}{\downarrow}_{\overline{B}} \sim S^{\bar{\alpha}} \oplus S^{\bar{\gamma}} \oplus S^{\bar{\epsilon}}; \\
\text{(F3)} & S^{\widetilde{\gamma}}{\uparrow}^{\overline{B}} \sim S^{\bar{\beta}} \oplus S^{\bar{\delta}} \oplus S^{\bar{\kappa}}; & S^{\gamma}{\downarrow}_{\overline{B}} \sim S^{\bar{\beta}} \oplus S^{\bar{\gamma}} \oplus S^{\bar{\kappa}}; \\
\text{(F4)} & S^{\widetilde{\delta}}{\uparrow}^{\overline{B}} \sim S^{\bar{\gamma}} \oplus S^{\bar{\epsilon}} \oplus S^{\bar{\kappa}}; & S^{\delta}{\downarrow}_{\overline{B}} \sim S^{\bar{\delta}} \oplus S^{\bar{\epsilon}} \oplus S^{\bar{\kappa}}.
\end{array}
$$

For each non-exceptional Specht module $S^{\tilde{\lambda}}$ of $\widetilde{B}$ with $S^{\tilde{\lambda}}{\uparrow}^{B} \sim 2S^{\lambda}$, there exist two Specht modules $S^{\bar{\lambda}}$ and $S^{\bar{\mu}}$ of $\overline{B}$ (with $\bar{\lambda} > \bar{\mu}$) such that $S^{\tilde{\lambda}}{\uparrow}^{\overline{B}} \sim S^{\bar{\lambda}} \oplus S^{\bar{\mu}} \sim S^{\lambda}{\downarrow}_{\overline{B}}$, $S^{\bar{\lambda}}{\downarrow}_{\widetilde{B}} \cong S^{\bar{\mu}}{\downarrow}_{\widetilde{B}} \cong S^{\tilde{\lambda}}$ and $S^{\bar{\lambda}}{\uparrow}^{B} \cong S^{\bar{\mu}}{\uparrow}^{B} \cong S^{\lambda}$.

Lemma 4.2. *Suppose $D^{\tilde{\lambda}}$ is a non-exceptional simple module of $\widetilde{B}$. Then $(D^{\tilde{\lambda}}{\uparrow}^{\overline{B}}){\downarrow}_{\widetilde{B}} \cong 2D^{\tilde{\lambda}}$.*

Similarly, if D^{λ} is a non-exceptional simple module of B, then $(D^{\lambda}{\downarrow}_{\overline{B}}){\uparrow}^{B} \cong 2D^{\lambda}$.

Proof. By the Branching Rule, we see that, for a non-exceptional Specht module $S^{\tilde{\lambda}}$ of $\widetilde{B}$, $(S^{\tilde{\lambda}}{\uparrow}^{\overline{B}}){\downarrow}_{\widetilde{B}} \sim 2S^{\tilde{\lambda}}$. Thus using induction, we can show that, for $\tilde{\lambda} > \widetilde{\alpha}$, $(D^{\tilde{\lambda}}{\uparrow}^{\overline{B}}){\downarrow}_{\widetilde{B}} \sim 2D^{\tilde{\lambda}}$, and hence that $(D^{\tilde{\lambda}}{\uparrow}^{\overline{B}}){\downarrow}_{\widetilde{B}} \cong 2D^{\tilde{\lambda}}$, since the simple modules of $\widetilde{B}$ do not self-extend. Now using relationships (F1), (E1)–(E3), we see $(S^{\widetilde{\alpha}}{\uparrow}^{\overline{B}}){\downarrow}_{\widetilde{B}} \sim 3S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}}$, so that $(D^{\widetilde{\alpha}}{\uparrow}^{\overline{B}}){\downarrow}_{\widetilde{B}} \sim S^{\widetilde{\alpha}} \oplus S^{\widetilde{\beta}} \oplus S^{\widetilde{\gamma}} \oplus S^{\widetilde{\delta}} \oplus 2D^{\widetilde{\alpha}}$. Noting that $D^{\widetilde{\alpha}}$ is a composition factor of $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$, and using relationships (E1)–(E6) and (F1)–(F4), we see that the lemma also holds for $\tilde{\lambda} \in \{\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$. Since the remaining Specht modules of $\widetilde{B}$ do not have $D^{\widetilde{\alpha}}$ as a composition factor, the lemma holds for all $\tilde{\lambda} \neq \widetilde{\alpha}$ inductively. An analogous argument applies to D^{λ}. $\square$

Proposition 4.3. *For each non-exceptional simple module D^{λ} of B, with $D^{\lambda}{\downarrow}_{\widetilde{B}} \cong 2D^{\widetilde{\lambda}}$ there exists a unique simple module $D^{\bar{\lambda}}$ of $\overline{B}$ such that*

(1) $D^{\bar{\lambda}}{\uparrow}^{B} \cong D^{\lambda}$;

(2) *$D^\lambda\downarrow_{\overline{B}}$ is non-simple, has head and socle both isomorphic to $D^{\bar\lambda}$, and all the composition factors $D^{\bar\mu}$ in its heart satisfy $D^{\bar\mu}\uparrow^B = 0$;*
(3) *$D^{\bar\lambda}\downarrow_{\widetilde{B}} \cong D^{\tilde\lambda}$;*
(4) *$D^{\tilde\lambda}\uparrow^{\overline{B}}$ is non-simple, has head and socle both isomorphic to $D^{\bar\lambda}$, and all the composition factors $D^{\bar\mu}$ in its heart satisfy $D^{\bar\mu}\downarrow_{\widetilde{B}} = 0$;*
(5) *$D^{\tilde\lambda}\uparrow^{\overline{B}} \cong D^\lambda\downarrow_{\overline{B}}$.*

Proof. Kleshchev's results on restricted simple modules [6] show that $D^\lambda\downarrow_{\overline{B}}$ has a simple head and simple socle which are isomorphic, to $D^{\bar\lambda}$ say. By Frobenius reciprocity, $D^{\bar\lambda}\uparrow^B$ has head and socle both isomorphic to D^λ. Using Lemma 4.2, we have $(D^\lambda\downarrow_{\overline{B}})\uparrow^B \cong 2D^\lambda$, so that $D^\lambda\downarrow_{\overline{B}}$ is not simple, as otherwise $(D^\lambda\downarrow_{\overline{B}})\uparrow^B$ will be indecomposable. Hence $D^{\bar\lambda}$ occurs at least twice in $D^\lambda\downarrow_{\overline{B}}$. Looking at Lemma 4.2 again, we can then conclude statements (1) and (2).

Now, Kleshchev's results on the socles of restricted simple modules [6] show that $D^{\tilde\lambda} \cong \mathrm{soc}(D^{\bar\lambda}\downarrow_{\widetilde{B}})$ and that $D^{\bar\lambda} \cong \mathrm{soc}(D^{\tilde\lambda}\uparrow^{\overline{B}})$. Similar arguments to those used in (1) and (2) show that (3) and (4) also hold.

For (5), using Lemma 4.1 and Frobenius reciprocity, we see that the dimension of $\mathrm{Hom}(D^\lambda\downarrow_{\overline{B}}, D^{\tilde\lambda}\uparrow^{\overline{B}}) \cong \mathrm{Hom}(D^\lambda, D^{\tilde\lambda}\uparrow^B)$ is 2. Since the multiplicities of $D^{\bar\lambda}$ as composition factors of $D^\lambda\downarrow_{\overline{B}}$ and $D^{\widetilde{\lambda}}\uparrow^{\overline{B}}$ are both 2, and they occur as the head and socle of both modules, this implies that there is an isomorphism from $D^\lambda\downarrow_{\overline{B}}$ to $D^{\tilde\lambda}\uparrow^{\overline{B}}$. $\square$

Corollary 4.4. *Let D^λ be a non-exceptional simple module of B, with $D^\lambda\downarrow_{\widetilde{B}} \cong 2D^{\tilde\lambda}$. If $\overline{M}$ is any $\overline{B}$-module, then, for any nonnegative integer r, we have*

$$\mathrm{Ext}^r\big(\overline{M}\uparrow^B, D^\lambda\big) \cong \mathrm{Ext}^r\big(\overline{M}\downarrow_{\widetilde{B}}, D^{\tilde\lambda}\big).$$

Proof. Since $D^\lambda\downarrow_{\overline{B}} \cong D^{\tilde\lambda}\uparrow^{\overline{B}}$, we have

$$\mathrm{Ext}^r\big(\overline{M}\uparrow^B, D^\lambda\big) \cong \mathrm{Ext}^r\big(\overline{M}, D^\lambda\downarrow_{\overline{B}}\big) \cong \mathrm{Ext}^r\big(\overline{M}, D^{\tilde\lambda}\uparrow^{\overline{B}}\big) \cong \mathrm{Ext}^r\big(\overline{M}\downarrow_{\widetilde{B}}, D^{\tilde\lambda}\big). \qquad \square$$

Lemma 4.5. *We have the following multiplicities involving exceptional simple modules*:

$$\big[D^\alpha\downarrow_{\overline{B}} : D^{\bar\alpha}\big] = 3; \qquad \big[D^{\widetilde{\alpha}}\uparrow^{\overline{B}} : D^{\bar\alpha}\big] = 3; \qquad \big[D^{\bar\alpha}\uparrow^B : D^\alpha\big] = 2; \qquad \big[D^{\bar\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}\big] = 2.$$

Hence, $D^{\bar\alpha}$ is a composition factor of $S^{\bar\alpha}$, $S^{\bar\beta}$, $S^{\bar\gamma}$, $S^{\bar\delta}$, $S^{\bar\epsilon}$ and $S^{\bar\kappa}$, all with multiplicity 1. *In no other Specht module can $D^{\bar\alpha}$ be a composition factor.*

Proof. Using Proposition 4.3 and relationship (E1), we see that the multiplicities $[D^{\bar\alpha}\uparrow^B : D^\alpha]$ and $[D^{\bar\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}]$ are as stated. Relationships (E2)–(E6) now tell us that $D^{\bar\alpha}$ occurs at most once in each of the exceptional Specht modules of $\overline{B}$. For example, $[S^{\bar\beta}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}] = 2$ by relationship (E2), and since $[D^{\bar\alpha}\downarrow_{\widetilde{B}} : D^{\widetilde{\alpha}}] = 2$, we see that $D^{\bar\alpha}$ can at most occur once in $S^{\bar\beta}$. Kleshchev's result on restricted simple modules [7] shows that $[D^\alpha\downarrow_{\overline{B}} : D^{\bar\alpha}] = 3$.

Thus, together with relationships (F1)–(F4), we see that $D^{\bar{\alpha}}$ must occur exactly once in each of the remaining exceptional Specht modules of $\overline{B}$. For example, relationship (F3) shows that $[S^{\bar{\beta}} \oplus S^{\bar{\gamma}} \oplus S^{\bar{\kappa}} : D^{\bar{\alpha}}] \geqslant 3$; this forces $D^{\bar{\alpha}}$ to occur exactly once in each of $S^{\bar{\beta}}$, $S^{\bar{\gamma}}$ and $S^{\bar{\kappa}}$. The last multiplicity $[D^{\widetilde{\alpha}}{\uparrow}^{\overline{B}} : D^{\bar{\alpha}}]$ can now be seen easily, say using (F1). Now, if $D^{\bar{\alpha}}$ is a composition factor of $S^{\bar{\lambda}}$, then $D^{\widetilde{\alpha}}$ is a composition factor of $S^{\bar{\lambda}}{\downarrow}_{\widetilde{B}}$. But if $S^{\bar{\lambda}}$ is not exceptional, then $S^{\bar{\lambda}}{\downarrow}_{\widetilde{B}} \cong S^{\tilde{\lambda}}$ is also not exceptional, and thus will not have $D^{\widetilde{\alpha}}$ as a composition factor. □

Proposition 4.6. *Let D^{λ} be a non-exceptional simple module of B, with $D^{\bar{\lambda}} \cong \operatorname{soc}(D^{\lambda}{\downarrow}_{\overline{B}})$ and $D^{\bar{\lambda}}{\downarrow}_{\widetilde{B}} \cong D^{\tilde{\lambda}}$. The following table gives a complete list of possibilities of multiplicities of D^{λ}, $D^{\bar{\lambda}}$ and $D^{\tilde{\lambda}}$ as composition factors of some modules which have simple head and simple socle both isomorphic to an exceptional simple module*:

$[P(D^{\widetilde{\alpha}}) : D^{\tilde{\lambda}}]$	$[P(D^{\bar{\alpha}}) : D^{\bar{\lambda}}]$	$[P(D^{\alpha}) : D^{\lambda}]$	$[D^{\widetilde{\alpha}}{\uparrow}^{\overline{B}} : D^{\bar{\lambda}}]$	$[D^{\bar{\alpha}}{\downarrow}_{\widetilde{B}} : D^{\tilde{\lambda}}]$	$[D^{\bar{\alpha}}{\uparrow}^{B} : D^{\lambda}]$	$[D^{\alpha}{\downarrow}_{\overline{B}} : D^{\bar{\lambda}}]$
0	0	0	0	0	0	0
1	3	3	1	0	1	0
2	6	2	2	0	0	2
3	3	1	0	1	0	1
4	6	4	1	1	1	1
4	12	4	4	0	0	4

Proof. Again, we rely heavily on relationships (E1)–(E6) and (F1)–(F4) in proving this proposition. We will only justify the entries in the second row. Similar arguments apply for all other rows. Since $[P(D^{\widetilde{\alpha}}) : D^{\tilde{\lambda}}] = 1$, there is exactly one Specht module among $S^{\widetilde{\alpha}}$, $S^{\widetilde{\beta}}$, $S^{\widetilde{\gamma}}$ and $S^{\widetilde{\delta}}$ of which $D^{\tilde{\lambda}}$ is a composition factor. Since $D^{\bar{\lambda}}{\downarrow}_{\widetilde{B}} \cong D^{\tilde{\lambda}}$, we see that the multiplicities of $D^{\bar{\lambda}}$ as a composition factor of $S^{\bar{\alpha}}$, $S^{\bar{\beta}}$, $S^{\bar{\gamma}}$, $S^{\bar{\delta}}$, $S^{\bar{\epsilon}}$ and $S^{\bar{\kappa}}$ are at most 1, using relationships (E1)–(E6). For ease of reference, we will assume that $D^{\tilde{\lambda}}$ is a composition factor $S^{\widetilde{\gamma}}$. Relationship (E1) then tells us that $D^{\tilde{\lambda}}$ is not a composition factor of $D^{\bar{\alpha}}{\downarrow}_{\widetilde{B}}$. Furthermore, (E1)–(E6) show that $D^{\bar{\lambda}}$ is a composition factor of $S^{\bar{\beta}}$, $S^{\bar{\delta}}$ and $S^{\bar{\kappa}}$, and is not a composition factor of $S^{\bar{\alpha}}$, $S^{\bar{\gamma}}$ and $S^{\bar{\epsilon}}$. This shows that $[P(D^{\bar{\alpha}}) : D^{\bar{\lambda}}] = 3$. Moreover, relationship (F1) yields $[D^{\widetilde{\alpha}}{\uparrow}^{\overline{B}} : D^{\bar{\lambda}}] = 1$. Proceeding in this way, using (F2), we find that D^{λ} is not a composition factor of S^{β}, and $D^{\bar{\lambda}}$ is not a composition factor of $D^{\alpha}{\downarrow}_{\overline{B}}$. Relationships (F1)–(F4) then show that D^{λ} is a composition factor of S^{α}, S^{γ} and S^{δ}. Hence, $[P(D^{\alpha}) : D^{\lambda}] = 3$. Relationship (E1) then shows that $[D^{\bar{\alpha}}{\uparrow}^{B} : D^{\lambda}] = 1$. □

Lemma 4.7. *The modules $D^{\bar{\alpha}}{\uparrow}^{B}$ and $D^{\bar{\alpha}}{\downarrow}_{\widetilde{B}}$ have a common Loewy length* 3.

Proof. Using Proposition 4.6 we see that the multiplicity of a non-exceptional simple module as a composition factor of $D^{\bar{\alpha}}{\uparrow}^{B}$ (or $D^{\bar{\alpha}}{\downarrow}_{\widetilde{B}}$) is at most 1. Since $D^{\bar{\alpha}}{\uparrow}^{B}$ and $D^{\bar{\alpha}}{\downarrow}_{\widetilde{B}}$ are both indecomposable, non-simple (by Lemma 4.5) and self-dual, this implies that they have a common Loewy length 3. □

5. [3 : 2]-pairs

We now turn our attention back to the blocks B and $\widetilde{B}$. With the information about the intermediate block $\overline{B}$ made available in the previous section, we are able to study the blocks B and $\widetilde{B}$ in more detail.

We begin with the observation that both $D^\alpha{\downarrow_{\widetilde{B}}}$ and $D^{\widetilde{\alpha}}{\uparrow^B}$ are direct sums of two isomorphic indecomposable modules.

Lemma 5.1. *The restricted simple module $D^\alpha{\downarrow_{\widetilde{B}}}$ is a direct sum of two isomorphic modules, each of which has head and socle isomorphic to $D^{\widetilde{\alpha}}$ and another copy of $D^{\widetilde{\alpha}}$ in its heart. Similarly, $D^{\widetilde{\alpha}}{\uparrow^B}$ is a direct sum of two isomorphic modules, each of which has head and socle isomorphic to D^α and another copy of $D^{\widetilde{\alpha}}$ in its heart.*

Proof. It is clear from Proposition 4.6, Lemma 4.5 and Frobenius reciprocity that, if $[P(D^\alpha) : D^\lambda] = 1$, then $P(D^\lambda){\downarrow_{\widetilde{B}}} \cong 2P(D^{\widetilde{\lambda}})$ (where $D^\lambda{\downarrow_{\widetilde{B}}} \cong 2D^{\widetilde{\lambda}}$), so that $D^\alpha{\downarrow_{\widetilde{B}}}$ is necessarily a direct sum of two isomorphic modules. Each summand is self-dual, has three copies of $D^{\widetilde{\alpha}}$, with only $D^{\widetilde{\alpha}}$ occurring in its head and socle. If it is indecomposable then it will have a simple head and a simple socle both isomorphic to $D^{\widetilde{\alpha}}$ and another copy of $D^{\widetilde{\alpha}}$ in its heart. If it is decomposable, then it must be a direct sum of $D^{\widetilde{\alpha}}$ and another module $\widetilde{M}$ say, which has a simple head and simple socle isomorphic to $D^{\widetilde{\alpha}}$. Since $[D^\alpha{\downarrow_{\widetilde{B}}} : D^{\widetilde{\lambda}}] = 4$ from Proposition 4.6, we see that $D^{\widetilde{\lambda}}$ occurs twice in $\widetilde{M}$. But $D^{\bar{\alpha}}$ is a submodule of $D^\alpha{\downarrow_{\overline{B}}}$, so that $D^{\bar{\alpha}}{\downarrow_{\widetilde{B}}}$ is a submodule of $D^\alpha{\downarrow_{\widetilde{B}}} \cong 2D^{\widetilde{\alpha}} \oplus 2\widetilde{M}$. Now, $D^{\bar{\alpha}}{\downarrow_{\widetilde{B}}}$ is indecomposable and has only a copy of $D^{\tilde{\lambda}}$, and thus cannot be a submodule of $\widetilde{M}$, which is a contradiction.

Finally it remains to show the existence of such a D^λ. Now, if δ is p-regular, then certainly D^δ only occurs once in $P(D^\alpha)$, while if δ is p-singular, then α' is p-regular, so that S^α has a simple socle which occurs exactly once in $P(D^\alpha)$.

Similar arguments hold for $D^{\widetilde{\alpha}}{\uparrow^B}$. $\square$

Notation. We shall denote the indecomposable direct summand of $D^\alpha{\downarrow_{\widetilde{B}}}$ by $\widetilde{L}_5$ and that of $D^{\widetilde{\alpha}}{\uparrow^B}$ by L_5.

It is clear that L_5 has Loewy length greater than or equal to 5, since D^α does not extend itself. Also L_5 has a submodule isomorphic to $D^{\bar{\alpha}}{\uparrow^B}$. Analogous statements also hold for $\widetilde{L}_5$.

Proposition 5.2. *The projective module $P(D^\alpha)$ has exactly* (*not just up to isomorphism*) *four submodules with simple head D^α, namely D^α, $D^{\bar{\alpha}}{\uparrow^B}$, L_5 and $P(D^\alpha)$ where the first three are embedded into $P(D^\alpha)$, and they are totally ordered by inclusion.*

Proof. Any proper submodule of $P(D^\alpha)$ with simple head D^α is contained in L_5 by Lemma 2.1 (with $M = \mathrm{rad}(P(D^\alpha))$). Thus D^α and $D^{\bar{\alpha}}{\uparrow^B}$ are contained properly in L_5. Applying Lemma 2.1 again with $M = \mathrm{rad}(L_5)$ then shows $D^\alpha \subset D^{\bar{\alpha}}{\uparrow^B}$. The submodules listed above are the only four such since $\dim \mathrm{End}(P(D^\alpha)) = 4$. $\square$

Similarly, viewing $D^{\widetilde{\alpha}}$, $D^{\bar{\alpha}}{\downarrow}_{\widetilde{B}}$, $\widetilde{L}_5$ as submodules of $P(D^{\widetilde{\alpha}})$, these are the only proper submodules of $P(D^{\widetilde{\alpha}})$ having head isomorphic to $D^{\widetilde{\alpha}}$, and they are totally ordered by inclusion.

Lemma 5.3. *The Loewy length of L_5 is at least 5, and that of $P(D^{\alpha})$ is at least 7. An analogous statement holds for $P(D^{\widetilde{\alpha}})$.*

Proof. Since there are no self-extensions, then the Loewy length of L_5 is at least 5, by Lemma 5.1. Also, viewing L_5 as a submodule of $P(D^{\alpha})$, the head of L_5 is lying in or below the third Loewy layer of $P(D^{\alpha})$. Hence $P(D^{\alpha})$ has Loewy length at least 7. □

Lemma 5.4.

(1) *$S^{\bar{\kappa}}{\uparrow}^{B}$ has simple socle D^{α} and, if γ and δ are p-regular, simple head D^{γ}.*
(2) *$S^{\widetilde{\delta}}{\uparrow}^{B}$ has socle $2D^{\alpha}$ and, if β is regular, head $2D^{\beta}$.*

Analogous statements hold for $S^{\bar{\kappa}}{\downarrow}_{\widetilde{B}}$ and $S^{\delta}{\downarrow}_{\widetilde{B}}$.

Proof. (1) Note that $S^{\bar{\kappa}}$ has a simple socle $D^{\bar{\alpha}}$, so that by Frobenius reciprocity and Proposition 4.3, $S^{\bar{\kappa}}{\uparrow}^{B}$ has a simple socle D^{α}. By the Branching Rule, $S^{\bar{\kappa}}{\uparrow}^{B}$ has a Specht filtration of the form

$$\begin{matrix} S^{\gamma} \\ S^{\delta} \end{matrix}.$$

If γ and δ are p-regular, then from the Specht filtration, we see that its head contains D^{γ}, and possibly D^{δ}, which we shall proceed to rule out. By Kleshchev's results on restricted simple modules, $D^{\bar{\kappa}}$ is a composition factor of $D^{\gamma}{\downarrow}_{\overline{B}}$, and relationship (F4) then shows that $D^{\bar{\kappa}}$ is not a composition factor of $D^{\delta}{\downarrow}_{\overline{B}}$. By Frobenius reciprocity, this implies that D^{δ} does not occur in the head of $S^{\bar{\kappa}}{\uparrow}^{B}$.

(2) This follows from Frobenius reciprocity, since $S^{\widetilde{\delta}}$ has a simple socle $D^{\widetilde{\alpha}}$ and, if β is regular, a simple head $\widetilde{\delta}$. □

Corollary 5.5. *Let $0 \subset M_3 \subset M_2 \subset M_1 \subset P(D^{\alpha})$ be the Specht filtration of $P(D^{\alpha})$.*

(1) *If δ is p-regular, then $M_3 = S^{\delta}$.*
(2) *If γ and δ are p-regular, then $M_2 = S^{\bar{\kappa}}{\uparrow}^{B}$.*
(3) *If β is p-regular, then $2M_1 = S^{\widetilde{\delta}}{\uparrow}^{B}$.*

Analogous statements hold for the Specht filtration of $P(D^{\widetilde{\alpha}})$.

In this corollary, we are viewing S^{δ} and $S^{\bar{\kappa}}{\uparrow}^{B}$ as submodules of $P(D^{\alpha})$, and $S^{\widetilde{\delta}}{\uparrow}^{B}$ as a submodule of $2P(D^{\alpha})$.

Proof. By Lemmas 2.1 and 5.4, if the respective p-regular conditions hold, then

$$S^{\delta} \subseteq M_3, \qquad S^{\bar{\kappa}}{\uparrow}^{B} \subseteq M_2, \qquad S^{\tilde{\delta}}{\uparrow}^{B} \subseteq 2M_1.$$

Equality holds for each instance, since the modules on both sides have Specht filtrations with the same factors (hence the same composition length). □

Corollary 5.6. *Suppose β is p-regular, and $\operatorname{Ext}^1(D^{\alpha}, D^{\lambda}) \neq 0$. Then $\lambda = \beta$ or D^{λ} occurs in the second Loewy layer of S^{α}.*

An analogous statement holds in the block $\widetilde{B}$.

Proof. By Lemma 5.4(2) and Corollary 5.5(3), M_1 has head D^{β}, and the statement follows immediately. □

Lemma 5.7. *Suppose the exceptional partitions are all p-regular. Then D^{γ} lies in or below the third Loewy layer of $P(D^{\alpha})$, and D^{δ} lies in or below the fourth Loewy layer of $P(D^{\alpha})$.*

An analogous statement holds for $P(D^{\tilde{\alpha}})$.

Proof. This follows from Lemma 5.4(1), Corollaries 5.5(2) and 5.6. □

We now give a sufficient condition for non-zero extensions between D^{α} (or $D^{\tilde{\alpha}}$) and a simple module.

Lemma 5.8. *Suppose $[P(D^{\alpha}) : D^{\lambda}] = 3$. Then $\operatorname{Ext}^1(D^{\alpha}, D^{\lambda}) \neq 0$.*

An analogous statement holds for $D^{\tilde{\alpha}}$.

Proof. By Lemma 4.7, $D^{\bar{\alpha}}{\uparrow}^{B}$ has Loewy length 3. By Proposition 4.6, we see that D^{λ} is a composition factor of $D^{\bar{\alpha}}{\uparrow}^{B}$, and hence lies in its semisimple heart. This shows that D^{α} extends D^{λ}. □

The following technical results will be required in the last section.

Lemma 5.9. *Suppose $P(D^{\alpha})$ has a subquotient with simple head D^{λ} and simple socle D^{μ}, where $[P(D^{\alpha}) : D^{\lambda}] = 2 = [P(D^{\alpha}) : D^{\mu}] + 1$. Then*

(1) $\operatorname{Ext}^1(D^{\bar{\alpha}}{\uparrow}^{B}, D^{\mu}) = 0$,
(2) $\operatorname{Ext}^1(D^{\tilde{\alpha}}, D^{\tilde{\lambda}}) = 0$,

where $D^{\lambda}{\downarrow}_{\widetilde{B}} \cong 2D^{\tilde{\lambda}}$.

Proof. Let $0 \subseteq N \subseteq M \subseteq P(D^{\alpha})$ such that M/N has simple head D^{λ} and simple socle D^{μ}.

(1) We may assume that M has simple head D^{λ}. By Proposition 4.6, D^{λ} does not occur in $D^{\bar{\alpha}}{\uparrow}^{B}$, so that $M \subseteq \Omega(D^{\bar{\alpha}}{\uparrow}^{B})$ by Lemma 2.1. Since D^{μ} does not lie in the head

of M, and hence of $\Omega(D^{\bar{\alpha}}\uparrow^B)$ (since $[\Omega(D^{\bar{\alpha}}\uparrow^B):D^\mu]=[M:D^\mu]$), we see that $\mathrm{Ext}^1(D^{\bar{\alpha}}\uparrow^B, D^\mu)=0$.

(2) We may assume that $P(D^\alpha)/N$ has simple socle D^μ. Note that D^μ does not occur in N and L_5, the latter by Proposition 4.6. Hence $L_5 \subseteq N$ by the dual version of Lemma 2.1. Since D^λ does not lie in the socle of $P(D^\alpha)/N$, and hence of $P(D^\alpha)/L_5$ (since $[P(D^\alpha)/L_5 : D^\lambda]=[P(D^\alpha)/N:D^\lambda]$), we see that

$$\mathrm{Ext}^1\bigl(2D^{\tilde{\lambda}}, D^{\widetilde{\alpha}}\bigr) \cong \mathrm{Ext}^1\bigl(D^\lambda, 2L_5\bigr)=0. \qquad \square$$

Similarly, if $P(D^{\widetilde{\alpha}})$ has a subquotient with simple head $D^{\tilde{\lambda}}$ and simple socle $D^{\widetilde{\mu}}$, where $[P(D^{\widetilde{\alpha}}):D^{\tilde{\lambda}}]=2=[P(D^{\widetilde{\alpha}}):D^{\widetilde{\mu}}]+1$, then $\mathrm{Ext}^1(D^{\bar{\alpha}}\downarrow_{\widetilde{B}}, D^{\widetilde{\mu}})=0=\mathrm{Ext}^1(D^\alpha, D^\lambda)=0$, where $D^{\tilde{\lambda}}\uparrow^B \cong 2D^\lambda$.

Lemma 5.10. *Suppose γ and δ are p-regular, and*

$$[S^\gamma : D^\lambda]=[S^\delta : D^\lambda]=1=\bigl[P(D^\alpha):D^\lambda\bigr]-2.$$

Then D^λ lies in or below the third Loewy layer of S^δ.

An analogous statement holds for $S^{\widetilde{\delta}}$.

Proof. By Lemma 5.4(1), $S^{\bar{\kappa}}\uparrow^B$ has a simple head D^γ and a simple socle D^α. Let M be the submodule of $P(D^\alpha)$ that is isomorphic to

$$\bigl(S^{\bar{\kappa}}\uparrow^B + \Omega(L_5)\bigr)/\Omega(L_5) \subseteq P(D^\alpha)/\Omega(L_5) \cong L_5 \subseteq P(D^\alpha).$$

Then M has a simple socle D^α and a simple head D^γ, since $M \cong S^{\bar{\kappa}}\uparrow^B/(\Omega(L_5)\cap S^{\bar{\kappa}}\uparrow^B)$. Furthermore, $[M:D^\lambda]>0$, as $[S^{\bar{\kappa}}\uparrow^B : D^\lambda]=2>1=[\Omega(L_5):D^\lambda]$ by Proposition 4.6. Viewing $S^{\bar{\kappa}}\uparrow^B$ as a submodule of $P(D^\alpha)$, we have $M \subseteq S^{\bar{\kappa}}\uparrow^B$ by Lemma 2.1. In fact, $M \nsubseteq S^{\bar{\kappa}}\uparrow^B$ so that $M \subseteq \mathrm{rad}(S^{\bar{\kappa}}\uparrow^B)$. Now, viewing S^δ as a submodule of $S^{\bar{\kappa}}\uparrow^B$, we see that $M \subseteq S^\delta$ by Lemma 2.1 (applied to $\mathrm{rad}(S^{\bar{\kappa}}\uparrow^B)$). The lemma thus follows. $\square$

6. Some sufficient conditions

In this section, we show that B inherits the properties that its Ext-quiver is bipartite and its principal indecomposable modules have a common Loewy length from $\widetilde{B}$ if the $[3:2]$-pair has the following conditions:

(Y1) If D^λ is a non-exceptional simple module of B, then $[P(D^\alpha):D^\lambda]\leqslant 3$.
(Y2) If $[P(D^\alpha):D^\lambda]$ is odd, then $\mathrm{Ext}^1(D^{\bar{\alpha}}\uparrow^B, D^\lambda)=0$.
(Y3) If $[P(D^\alpha):D^\lambda]=2$, with $D^\lambda\downarrow_{\widetilde{B}} \cong 2D^{\tilde{\lambda}}$, then $\mathrm{Ext}^1(D^\alpha, D^\lambda)=0$ if and only if $\mathrm{Ext}^1(D^{\widetilde{\alpha}}, D^{\tilde{\lambda}})=0$.

While these conditions appear to be imposed on the block B, they are equivalent to the analogous conditions imposed on the block $\widetilde{B}$, which can be seen using Proposition 4.6 and Corollary 4.4.

Remark. If (Y1) holds, then

(1) the last two possibilities in Proposition 4.6 do not occur;
(2) $[L_5 : D^\lambda] = [D^{\bar{\alpha}}\uparrow^B : D^\lambda] + 1$ for all composition factors D^λ of L_5;
(3) $[\widetilde{L}_5 : D^{\tilde{\lambda}}] = [D^{\bar{\alpha}}\downarrow_{\widetilde{B}} : D^{\tilde{\lambda}}] + 1$ for all composition factors $D^{\tilde{\lambda}}$ of $\widetilde{L}_5$;
(4) $[P(D^\alpha) : D^\mu] = [L_5 : D^\mu] + 1$ for all composition factors D^μ of $P(D^\alpha)$;
(5) $[P(D^{\widetilde{\alpha}}) : D^{\widetilde{\mu}}] = [\widetilde{L}_5 : D^{\widetilde{\mu}}] + 1$ for all composition factors $D^{\widetilde{\mu}}$ of $P(D^{\widetilde{\alpha}})$.

We now proceed to extract more information about [3 : 2]-pairs satisfying one or more conditions. We will merely state and prove the results for B, and it is not difficult to see that analogous results and proofs also hold for $\widetilde{B}$, usually by interchanging D^α with $D^{\widetilde{\alpha}}$, L_5 with $\widetilde{L}_5$, D^λ with $D^{\tilde{\lambda}}$, restriction with induction, and B with $\widetilde{B}$.

Lemma 6.1. *Suppose* (Y1) *holds. Then L_5 has Loewy length* 5.

Proof. The copy of D^α lying in the heart of L_5 must lie in the third Loewy layer of L_5, since $D^{\bar{\alpha}}\uparrow^B$ is a quotient of L_5 and has Loewy length 3. Let L_3 be the submodule of L_5 isomorphic to $D^{\bar{\alpha}}\uparrow^B$. The head of L_3, being the copy of D^α lying in the heart of L_5, lies in the third Loewy layer of L_5. Each composition factor in the heart of L_3 occurs twice in L_5, once in its second Loewy layer and once below the third Loewy layer. Since each composition factor of L_5 which does not occur in L_3 only occurs once in L_5, and L_5 is self-dual, this implies that it must lie in the second or third Loewy layer of L_5. Thus L_5 has Loewy length 5. $\square$

Out of the proof we get an immediate corollary.

Corollary 6.2. *Suppose* (Y1) *holds and* $[L_5 : D^\lambda] = 2$. *Then D^λ occurs once each in the second and fourth Loewy layers of L_5.*

Since $P(D^\alpha)$ has a quotient isomorphic to L_5, the two copies of D^α lying in the heart of $P(D^\alpha)$ lies in the third and fifth Loewy layer of $P(D^\alpha)$, by Lemma 6.1. Now, viewing L_5 as a submodule of $P(D^\alpha)$, we see that head of L_5 lies in the third Loewy layer of $P(D^\alpha)$ and the copy of D^α in its heart lies in the fifth Loewy layer of $P(D^\alpha)$. Each composition factor occurring twice in L_5 occurs thrice in $P(D^\alpha)$, in the second, the fourth and below the fifth Loewy layers of $P(D^\alpha)$. Each composition factor occurring once in L_5 occurs twice in $P(D^\alpha)$, once in the second or third Loewy layer and once in or below the fourth Loewy layer. Each composition factor of $P(D^\alpha)$ which is not a composition factor of L_5 occurs only once. Together with the fact that $P(D^\alpha)$ is self-dual, we have:

Proposition 6.3. *Suppose* (Y1) *holds. Then the Loewy length of $P(D^\alpha)$ is* 7.

Corollary 6.4. *Suppose* (Y1) *holds and* $[P(D^\alpha) : D^\lambda] = m$.

(1) *If* $m = 3$, *then* D^λ *occurs once each in the second, fourth, and sixth Loewy layers of* $P(D^\alpha)$.
(2) *If* $m = 2$, *then* D^λ *occurs once in the second or third Loewy layer, and once in the fourth or fifth Loewy layer of* $P(D^\alpha)$.
(3) *If* $m = 1$, *then* D^λ *occurs in the second, third or fourth Loewy layer of* $P(D^\alpha)$.

Also, D^α *occurs once each in the first, third, fifth and seventh Loewy layers of* $P(D^\alpha)$.

Lemma 6.5. *Suppose* (Y2) *holds, and* $[P(D^\alpha) : D^\lambda] = 1$. *Then* $\mathrm{Ext}^1(D^\alpha, D^\lambda) = 0$.

Proof. By (Y2), we have $\mathrm{Ext}^1(D^{\bar\alpha}{\uparrow}^B, D^\lambda) = 0$. By Proposition 4.6, $[D^{\bar\alpha}{\uparrow}^B : D^\lambda] = 0$. The result thus follows. □

Corollary 6.6. *Suppose* (Y1) *and* (Y2) *hold, and* $[P(D^\alpha) : D^\lambda] = 1$. *Then* D^λ *lies in the third or fourth Loewy layer of* $P(D^\alpha)$.

Condition (Y3) allows us to refine this corollary.

Lemma 6.7. *Suppose* (Y1)–(Y3) *hold, and* $[P(D^\alpha) : D^\mu] = 1$. *Then* D^μ *lies in the fourth Loewy layer of* $P(D^\alpha)$.

Proof. If D^μ lies in the third Loewy layer of $P(D^\alpha)$, then there exists a quotient M of $P(D^\alpha)$ having a simple socle D^μ and Loewy length 3. There is at least one composition factor D^λ of the heart of M which has multiplicity 2 in $P(D^\alpha)$ (otherwise, all composition factors of the heart of M multiplicity 3 by Lemma 6.5, and they all occur in the heart of $D^{\bar\alpha}{\uparrow}^B$ by Proposition 4.6; the existence of M then implies that $\mathrm{Ext}^1(D^{\bar\alpha}{\uparrow}^B, D^\mu) \neq 0$, contradicting (Y2)). By self-duality of $P(D^\alpha)$ and Lemma 5.9(2), $\mathrm{Ext}^1(D^{\widetilde\alpha}, D^{\tilde\lambda}) = 0$ (where $D^\lambda{\downarrow}_{\widetilde B} \cong 2D^{\tilde\lambda}$). But this contradicts (Y3), since D^λ lies in the second Loewy layer of M, and hence of $P(D^\alpha)$. □

We shall now state the main theorem of this section.

Theorem 6.8. *Suppose* (Y1)–(Y3) *hold, and* $\widetilde B$ *has the properties that its* Ext*-quiver is bipartite and its principal indecomposable modules have a common Loewy length* 7. *Then* B *inherits these properties from* $\widetilde B$.

The remainder of this section will be devoted to proving the above theorem. We assume that (Y1)–(Y3) hold, and $\widetilde B$ has the properties that its Ext-quiver is bipartite and its principal indecomposable modules have a common Loewy length 7.

Let $\{\widetilde{\Lambda}_1, \widetilde{\Lambda}_2\}$ be a partition of the simple modules of $\widetilde{B}$ displaying the bipartite nature of the Ext-quiver of $\widetilde{B}$, with $D^{\widetilde{\alpha}} \in \widetilde{\Lambda}_1$. For $j \in \{1, 2\}$, define a subset Λ_j of simple modules of B by

$$\Lambda_j = \left\{ D^\lambda \mid \operatorname{soc}\left(D^{\tilde{\lambda}}\uparrow^B\right) \cong 2D^\lambda \text{ for some } \tilde{\lambda} \in \widetilde{\Lambda}_j \right\}.$$

Proposition 6.9. *The partition* $\{\Lambda_1, \Lambda_2\}$ *defined above displays the bipartite nature of the* Ext-*quiver of* B.

Proof. If suffices to show that D^α does not extend any simple module contained in Λ_1. By Corollaries 6.4 and 6.6, D^α extends D^μ only if $[P(D^\alpha) : D^\mu] = 3$ or 2 (equivalently $[P(D^{\widetilde{\alpha}}) : D^{\widetilde{\mu}}] = 1$ or 2). The analogue of Lemma 6.7 shows that those simple modules of the first case are contained in Λ_2, while (Y3) ensures the simple modules of the second case are also contained in Λ_2. □

Proposition 6.10. *Assume that the hypotheses of Theorem* 6.8 *hold. Let* $[P(D^\alpha) : D^\lambda] = 3$. *Then* $P(D^\lambda)$ *has Loewy length* 7.

Proof. Let $D^\lambda\downarrow_{\widetilde{B}} \cong 2D^{\tilde{\lambda}}$. By Proposition 4.6, $D^{\widetilde{\alpha}}$ occurs once in $P(D^{\tilde{\lambda}})$, so it is lying in the fourth Loewy layer of $P(D^{\tilde{\lambda}})$ by Lemma 6.7. For $2 \leqslant j \leqslant 6$, let $\widetilde{M}_j$ be the direct sum of simple modules lying in the jth Loewy layer of $P(D^{\tilde{\lambda}})$ and not isomorphic to $D^{\widetilde{\alpha}}$, and let M_j be such that $\widetilde{M}_j\uparrow^B \cong 2M_j$. Also, for $2 \leqslant l \leqslant 4$, let N_l be the direct sum of simple modules lying in the lth Loewy layer of L_5 and not isomorphic to D^α. By Frobenius reciprocity, $P(D^{\tilde{\lambda}})\uparrow^B \cong 2P(D^\lambda)$, and Corollary 6.4 tells us that D^α lies in the second, fourth and sixth Loewy layers of $P(D^\lambda)$. We have

$$P\left(D^{\tilde{\lambda}}\right) = \begin{matrix} D^{\tilde{\lambda}} \\ \widetilde{M}_2 \\ \widetilde{M}_3 \\ D^{\widetilde{\alpha}} \oplus \widetilde{M}_4 \\ \widetilde{M}_5 \\ \widetilde{M}_6 \\ D^{\tilde{\lambda}} \end{matrix} \quad \text{and} \quad P\left(D^\lambda\right) = \begin{matrix} D^\lambda \\ M_2 \\ M_3 \\ L_5 \oplus M_4 \\ M_5 \\ M_6 \\ D^\lambda \end{matrix} = \begin{matrix} D^\lambda \\ D^\alpha \oplus M_2 \\ N_2 \oplus M_3 \\ D^\alpha \oplus N_3 \oplus M_4 \\ N_4 \\ D^\alpha \\ M_5 \\ M_6 \\ D^\lambda \end{matrix}.$$

Now, if the Loewy length of $P(D^\lambda)$ is greater than 7, then by the bipartite nature of the Ext-quiver of B, it has to be 9, with some composition factor of M_5 lying in the seventh Loewy layer, directly below the copy of D^α in the sixth Loewy layer. But this will mean that there is no copy of D^α lying in the second socle layer, contradicting the self-duality of $P(D^\lambda)$. Hence $P(D^\lambda)$ has Loewy length 7. □

Proposition 6.11. *Assume that the hypotheses of Theorem* 6.8 *hold. Let* $[P(D^\alpha) : D^\lambda] = 1$. *Then* $P(D^\lambda)$ *has Loewy length* 7.

Proof. Using Lemma 6.7, we see that D^α is lying in the fourth Loewy layer of $P(D^\lambda)$. By self-duality of $P(D^\lambda)$, we see that its Loewy length is at least 7. Suppose its Loewy length is greater than 7; then it is at least 9, by the bipartite nature of the Ext-quiver of B. Thus there exists a submodule M of $P(D^\lambda)$ having a simple head (isomorphic to D^μ say) and the Loewy length at least 7. If M has no composition factor isomorphic to D^α, then $M\downarrow_{\widetilde{B}}$ is a $\widetilde{B}$-module having no projective summand and Loewy length at least 7, which is impossible. Hence M has a composition factor isomorphic to D^α. In fact, looking at the Loewy structure of M, we find that D^α has to lie in the second Loewy layer of M, so that D^μ necessarily extends D^α. Proposition 6.10 now tells us that $[P(D^\alpha):D^\mu]=2$. But this implies that $P(D^\lambda)$ has a quotient N (having Loewy length 4) which has a simple socle D^α and a copy of D^μ in its third Loewy layer. Since N is isomorphic to a submodule of $P(D^\alpha)$ and D^λ lies in the fourth Loewy layer of $P(D^\alpha)$, this further shows that there is a copy of D^μ in the sixth Loewy layer of $P(D^\alpha)$. But this is impossible, in view of Corollary 6.4. □

We need the next two lemmas to prove the next proposition.

Lemma 6.12. *Suppose* (Y1) *holds. Let M be a B-module with simple head D^α and simple socle D^λ, and having Loewy length* $\geqslant 4$, *where* $[P(D^\alpha):D^\lambda]=2$. *Then D^α is a composition factor of the heart of M.*

Proof. We show $[M:D^\alpha]\geqslant 2$, and for this, it suffices to show $L_5 \not\subseteq \Omega(M)$, or equivalently, M is not a quotient of $P(D^\alpha)/L_5$.

We know that D^λ occurs exactly once in the first three Loewy layers of $P(D^\alpha)$ by Corollary 6.4(2), and that $L_5 \subseteq \operatorname{rad}^2(P(D^\alpha))$ since D^α does not self-extend. Thus, as we also have $[P(D^\alpha)/L_5 : D^\lambda]=1$, we see that D^λ occurs in the first three Loewy layers of $P(D^\alpha)/L_5$, and so D^λ may only occur in the first three Loewy layers of any quotient of $P(D^\alpha)/L_5$. Hence, M is not a quotient of $P(D^\alpha)/L_5$. □

Lemma 6.13. *Suppose* (Y1) *holds. Let M be a submodule of $P(D^\alpha)$ having a simple head D^λ and Loewy length* 3, *where* $[P(D^\alpha):D^\lambda]=2$. *Then* $[M:D^\mu]=0$ *for all* D^μ $(\mu\neq\lambda)$ *such that* $[P(D^\alpha):D^\mu]=2$.

Proof. We first show that M is a submodule of L_5. Otherwise, D^λ occurs twice in $M+L_5$. Since D^λ occurs in the first three socle layers of both M and L_5, the two copies of D^λ are found in the first three socle layers of $M+L_5$. But this contradicts Corollary 6.4.

Thus, M is a submodule of L_5. Now, since L_5 is self-dual and $[L_5:D^\lambda]=1=[L_5:D^\mu]$, we see that M cannot have a composition factor isomorphic to D^μ. □

Proposition 6.14. *Assume that the hypotheses of Theorem* 6.8 *holds. Let*

$$\left[P(D^\alpha):D^\lambda\right]=2.$$

Then $P(D^\lambda)$ has Loewy length 7.

Proof. We have two cases to consider:

(1) $\mathrm{Ext}^1(D^\alpha, D^\lambda) = 0$.
(2) $\mathrm{Ext}^1(D^\alpha, D^\lambda) \neq 0$.

In the first case, D^α lies in the third and fifth Loewy layers of $P(D^\lambda)$. By self-duality of $P(D^\lambda)$, we see that its Loewy length is at least 7. Suppose $P(D^\lambda)$ has Loewy length greater than 7. Then it has a submodule M whose head (isomorphic to D^μ say) lies in the second Loewy layer of $P(D^\lambda)$ and whose Loewy length is at least 7. Using Propositions 6.10 and 6.11, we conclude that $[P(D^\alpha) : D^\mu] = 2$. Also, $\mathrm{Ext}^1(D^\alpha, D^\mu) \neq 0$. Now D^α must occur as a composition factor of M, otherwise $M{\downarrow_{\widetilde{B}}}$ would be a $\widetilde{B}$-module having no projective summand and Loewy length at least 7, which is impossible. By considering the Loewy structure of M, we see that, in fact, D^α lies in the second and/or fourth Loewy layers of M. If D^α lies in the second Loewy layer of M, we will have a quotient N of $P(D^\lambda)$ having a simple socle D^α, a copy of D^μ in its heart and Loewy length 3. But this is impossible by Lemma 6.13. On the other hand, if D^α lies in the fourth but not in the second Loewy layer of M, we will have a quotient N' of M having a simple head D^μ, a simple socle D^α, a heart which does not have D^α as a composition factor and Loewy length 4. But N' is isomorphic to a submodule of $P(D^\alpha)$ which is again impossible by Lemma 6.12. Thus, the Loewy length of $P(D^\lambda)$ is 7.

In the second case, the composition factors in the second Loewy layer of $P(D^\lambda)$ does not extend D^α by Proposition 6.9. Thus $\mathrm{rad}(P(D^\lambda))$ has Loewy length at most 6, by Proposition 6.11 and the first case of this proof, so that $P(D^\lambda)$ has Loewy length at most 7. Suppose for a contradiction that the Loewy length of $P(D^\lambda)$ is less than 7. It must then be 5, with a copy of D^α each occurring in its second and fourth Loewy layers. Let M be a quotient of $P(D^\lambda)$ having a simple socle D^α and Loewy length 4. Using Lemma 6.12, we see that M has another copy of D^α lying in its heart. Hence M has the following Loewy structure:

$$\begin{array}{c} D^\lambda \\ D^\alpha \oplus N_2 \\ N_3 \\ D^\alpha \end{array}.$$

Now, $M{\downarrow_{\widetilde{B}}}$ has head isomorphic to $2D^{\widetilde{\alpha}} \oplus 2D^{\tilde{\lambda}}$ and socle isomorphic to four copies of $D^{\widetilde{\alpha}}$ by Frobenius reciprocity. Since $P(D^\lambda){\downarrow_{\widetilde{B}}} \cong 2P(D^{\widetilde{\alpha}}) \oplus 2P(D^{\tilde{\lambda}})$, we see that $M{\downarrow_{\widetilde{B}}}$ is, in fact, a direct sum $2P(D^{\widetilde{\alpha}}) \oplus 2\widetilde{M}$, where $\widetilde{M}$ has head isomorphic to $D^{\tilde{\lambda}}$ and socle isomorphic to $D^{\widetilde{\alpha}}$, and has another copy of $D^{\widetilde{\alpha}}$ lying in its heart. Since $\widetilde{M}$ is a submodule of $P(D^{\widetilde{\alpha}})$, we can then conclude that $\widetilde{M}$ has Loewy length 4. In fact, if $N_j{\downarrow_{\widetilde{B}}} \cong 2\widetilde{N}_j$ for $j \in \{2, 3\}$, $\widetilde{M}$ has the following Loewy structure:

$$\begin{array}{c} D^{\tilde{\lambda}} \\ D^{\widetilde{\alpha}} \oplus \widetilde{N}_2 \\ \widetilde{N}_3' \\ D^{\widetilde{\alpha}} \end{array},$$

where $\widetilde{N}_3'$ has a submodule isomorphic to that of $\widetilde{N}_3$. Let $\Omega(M)$ have the following socle structure:

$$\begin{matrix} T_4 \\ T_3 \\ T_2 \\ D^\lambda \end{matrix}$$

(so T_j, which may be zero, lies in the jth socle layer of $P(D^\lambda)$). It is not difficult to see that $\Omega(M)\downarrow_{\widetilde{B}} \cong 2\Omega(\widetilde{M})$. Let $T_j\downarrow_{\widetilde{B}} \cong 2\widetilde{T}_j$. Then $\widetilde{T}_j$ is the jth socle layer of $\Omega(\widetilde{M})$. If T_4 and hence $\widetilde{T}_4$ are both non-zero, then the latter must in fact lie in the second Loewy layer of $P(D^{\tilde{\lambda}})$. Hence, regardless whether $\widetilde{T}_4$ is zero or not, since $P(D^{\tilde{\lambda}})$ has Loewy length 7 and the Ext-quiver of $\widetilde{B}$ is bipartite, we must have the socle of $\widetilde{M}$ extending a composition factor of $\widetilde{T}_3$. But this contradicts the self-duality of $P(D^{\tilde{\lambda}})$. $\square$

Thus, the main theorem of this section (Theorem 6.8) follows from Propositions 6.9–6.11 and 6.14.

7. Examples

We conclude this paper by verifying the sufficient conditions listed in the last section for certain $[3:2]$-pairs.

Let B_i $(1 \leqslant i \leqslant p)$ be the defect 3 block of $k\mathfrak{S}_{4p+2i-3}$ with p-core $(p+i-2, i-1)$. We use the $\langle\,\rangle$-notation with $3p+2$ beads to denote the partitions belonging to these blocks, and to avoid confusion, we include a subscript i to indicate that the partition belongs to B_i. For example, $\langle 1,2\rangle_2$ is the partition of B_2 whose abacus display of $3p+2$ beads has a bead of weight 2 on the first column, and a bead of weight 1 on second column.

The block B_1 is the principal block of $k\mathfrak{S}_{4p-1}$, and this block is shown in [10] to have the properties that its Ext-quiver is bipartite and its principal indecomposable modules have a common Loewy length 7. The blocks B_i and B_{i-1} $(2 \leqslant i \leqslant p)$ form a $[3:2]$-pair, and we denote the exceptional partition with respect to this $[3:2]$-pair by α_i, β_i, γ_i, δ_i, $\widetilde{\alpha}_i$, $\widetilde{\beta}_i$, $\widetilde{\gamma}_i$ and $\widetilde{\delta}_i$ (again, the subscript i is included to avoid confusion). We note that these exceptional partitions are all p-regular, except for β_2 and $\widetilde{\delta}_2$. Furthermore, α_i' is also p-regular, except for α_p'.

Proposition 7.1. *The* $[3:2]$*-pair* B_i *and* B_{i-1} *satisfy* (Y1)–(Y3).

As a corollary, we have

Theorem 7.2. *The blocks* B_i $(1 \leqslant i \leqslant p)$ *have the properties that their* Ext*-quivers are bipartite, and their principal indecomposable modules have a common Loewy length* 7.

The remainder of this paper will be devoted to proving Proposition 7.1.

Firstly, by calculating the decomposition matrix of B_i using Schaper's formula, we can easily verify that (Y1) is satisfied.

Now, by Proposition 4.6, if $[P(D^{\alpha_i}) : D^\lambda] > 0$, and $D^\lambda{\downarrow}_{B_{i-1}} \cong 2D^{\tilde{\lambda}}$, then $[P(D^{\alpha_i}) : D^\lambda] + [P(D^{\widetilde{\alpha}_i}) : D^{\tilde{\lambda}}] = 4$.

By Corollary 4.4, we have $\mathrm{Ext}^1(D^{\bar{\alpha}}{\uparrow}^B, D^\lambda) \cong \mathrm{Ext}^1(D^{\bar{\alpha}}{\downarrow}_{\widetilde{B}}, D^{\tilde{\lambda}})$, where $D^\lambda{\downarrow}_{\widetilde{B}} \cong 2D^{\tilde{\lambda}}$. Thus, to verify (Y2), it suffices to show that $\mathrm{Ext}^1(D^{\bar{\alpha}_i}{\uparrow}^{B_i}, D^\lambda) = 0$ whenever $[P(D^{\alpha_i}) : D^\lambda] = 1$, and $\mathrm{Ext}^1(D^{\bar{\alpha}_i}{\downarrow}_{B_{i-1}}, D^{\widetilde{\mu}}) = 0$ whenever $[P(D^{\widetilde{\alpha}_i}) : D^{\widetilde{\mu}}] = 1$.

Let $[P(D^{\alpha_i}) : D^\lambda] = 1$. From the decomposition matrix of B_i, we see $[S^{\beta_i} : D^\lambda] = 0$. If $[S^{\gamma_i} : D^\lambda]$ or $[S^{\delta_i} : D^\lambda] = 1$, then $[S^{\bar{\kappa}_i}{\uparrow}^{B_i} : D^\lambda] = 1$ by Corollary 5.5(2). Thus, $\mathrm{Ext}^1(D^{\bar{\alpha}_i}{\uparrow}^{B_i}, D^\lambda) = 0$ by Lemmas 5.4(1) and 5.9(1). If $[S^{\alpha_i} : D^\lambda = 1]$, then $i \leqslant p-1$, and D^λ is the socle of S^{α_i}. In this case, since $[S^{\alpha_i} : D^{\widetilde{\beta}_{i+1}}] = 1 = [P(D^{\alpha_i}) : D^{\widetilde{\beta}_{i+1}}] - 1$, $\mathrm{Ext}^1(D^{\bar{\alpha}_i}{\uparrow}^{B_i}, D^\lambda) = 0$ by Lemma 5.9(1).

Now let $[P(D^{\widetilde{\alpha}_i}) : D^{\widetilde{\mu}}] = 1$. From the decomposition matrix of B_{i-1}, we see that either

- $[S^{\widetilde{\gamma}_i} : D^{\widetilde{\mu}}] = 1$; or
- $[S^{\widetilde{\delta}_i} : D^{\widetilde{\mu}}] = 1$ and $i \geqslant 3$.

In both of these cases, $\mathrm{Ext}^1(D^{\bar{\alpha}_i}{\downarrow}_{B_{i-1}}, D^{\widetilde{\mu}}) = 0$ by the analogues of Lemmas 5.4(1) and 5.9(1).

For (Y3), we show a stronger version—if $[P(D^{\alpha_i}) : D^\lambda] = 2$, then $\mathrm{Ext}^1(D^{\alpha_i}, D^\lambda) = 0 = \mathrm{Ext}^1(D^{\widetilde{\alpha}_i}, D^{\tilde{\lambda}})$, where $D^\lambda{\downarrow}_{B_{i-1}} \cong 2D^{\tilde{\lambda}}$. By Corollary 5.6, we only need to look at $D^{\widetilde{\mu}}$ with $[P(D^{\widetilde{\alpha}_i}) : D^{\widetilde{\mu}}] = 2 = [S^{\widetilde{\alpha}_i} : D^{\widetilde{\mu}}] + 1$, and if $i \geqslant 3$, D^λ with $[P(D^{\alpha_i}) : D^\lambda] = 2 = [S^{\alpha_i} : D^\lambda] + 1$. Note that we can (and shall) assume that the Ext-quiver of B_{i-1} is bipartite when we are determining if $D^{\widetilde{\alpha}_i}$ could extend $D^{\widetilde{\mu}}$.

If $[P(D^{\widetilde{\alpha}_i}) : D^{\widetilde{\mu}}] = 2 = [S^{\widetilde{\alpha}_i} : D^{\widetilde{\mu}}] + 1$, then either

- $[S^{\widetilde{\beta}_i} : D^{\widetilde{\mu}}] = 1$ or $[S^{\widetilde{\gamma}_i} : D^{\widetilde{\mu}}] = 1$; or
- $[S^{\widetilde{\delta}_i} : D^{\widetilde{\mu}}] = 1$.

In the first case, relationships (D1) and (D3) and Proposition 4.6 show $[S^{\alpha_i} : D^\mu] = 1$, where $D^{\widetilde{\mu}}{\uparrow}^B \cong 2D^\mu$. As S^{α_p} is simple, isomorphic to D^{α_p} (from the decomposition matrix of B_p), we see that this case does not occur for $i = p$. Thus $i < p$, so that α_i' is p-regular and S^{α_i} has a simple socle, and hence $\mathrm{Ext}^1(D^{\widetilde{\alpha}_i}, D^{\widetilde{\mu}}) = 0$ by Lemma 5.9(2). In the second case, there is a copy of $D^{\widetilde{\mu}}$ lying in the fifth Loewy layer of $P(D^{\widetilde{\alpha}_i})$, by Lemma 5.7 and Corollary 6.4. By induction the Ext-quiver of B_{i-1} is bipartite so $\mathrm{Ext}^1(D^{\widetilde{\alpha}_i}, D^{\widetilde{\mu}}) = 0$.

If $[P(D^{\alpha_i}) : D^\lambda] = 2 = [S^{\alpha_i} : D^\lambda] + 1$, then from the decomposition matrix of B_i, we see that $i < p$ and $\lambda = \widetilde{\beta}_{i+1}$ or $\langle i, i+1\rangle_i$ or, if $i = p-2$, $\langle i\rangle_i$. Moreover, $[S^{\widetilde{\gamma}_{i+1}} : D^\lambda] = [S^{\widetilde{\delta}_{i+1}} : D^\lambda] = 1 = [P(D^{\widetilde{\alpha}_{i+1}}) : D^\lambda] - 2$. Thus D^λ lies in or below the third Loewy layer of $S^{\widetilde{\delta}_{i+1}} = S^{\alpha_i}$ by Lemma 5.10. Hence $\mathrm{Ext}^1(D^{\alpha_i}, D^\lambda) = 0$ by Corollary 5.6 for $i > 2$. For $i = 2$, S^{β_2} is simple, isomorphic to D^{α_2}. Thus if $\mathrm{Ext}^1(D^{\alpha_2}, D^\nu) \neq 0$, then either $\nu = \gamma_2$ or D^ν occurs in the second Loewy layer of S^{α_2} by Lemma 5.4(1) and Corollary 5.5(2). Hence, $\mathrm{Ext}^1(D^{\alpha_i}, D^\lambda) = 0$ for $i = 2$ as well. Finally, we show that $\mathrm{Ext}^1(D^{\alpha_2}, D^{\gamma_2}) = 0$. This is because $[S^{\widetilde{\gamma}_2} : D^{\langle 1,1,3\rangle_1}] = 1 = [P(D^{\widetilde{\alpha}_2}) : D^{\langle 1,1,3\rangle_1}]$, so that the analogue of Lemma 5.9(2) applies.

This completes the proof of Proposition 7.1.

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